Wave-Number Spectrum of Drift-Wave Turbulence

Ö. D. Gürcan,[1,*] X. Garbet,[2] P. Hennequin,[1] P. H. Diamond,[3] A. Casati,[2] and G. L. Falchetto[2]
[1]*Laboratoire de Physique des Plasmas, Ecole Polytechnique, CNRS, 91128 Palaiseau Cedex, France*
[2]*CEA, IRFM, F-13108 Saint Paul Lez Durance, France*
[3]*Center for Astrophysics and Space Sciences, University of California San Diego, 9500 Gilman Drive, La Jolla, California 92093-0424, USA*
(Received 19 December 2008; published 23 June 2009)

A simple model for the evolution of turbulence fluctuation spectra, which includes neighboring interactions leading to the usual dual cascade as well as disparate scale interactions corresponding to refraction by large scale structures, is derived. The model recovers the usual Kraichnan-Kolmogorov picture in the case of exclusively local interactions and midrange drive. On the other hand, when disparate scale interactions are dominant, a simple spectrum for the density fluctuations of the form $|n_k|^2 \propto k^{-3}/(1+k^2)^2$ is obtained. This simple prediction is then compared to, and found to be in fair agreement with, Tore Supra CO_2 laser scattering data.

DOI: 10.1103/PhysRevLett.102.255002 PACS numbers: 52.35.Ra

The wave-number spectrum is one of the few quantities that can be measured in a tokamak, which allows probing the characteristics of underlying microturbulence [1–4]. It is important for validation as a higher order observable (in contrast to heat flux and other scalar quantities) [5]. Today, direct numerical simulations of gyrokinetic Vlasov equations are the main tools for studying anomalous transport. Such simulations are used for predictions to experimentally inadmissible parameter regimes, which influence design decisions. It is crucial that these numerical simulations describe the same kind of microturbulence as observed in tokamaks. One of the ways this can be verified is by studying the wave-number spectrum of density fluctuations. However, a direct comparison between experiment and numerical simulation may not always be possible due to various underlying assumptions in simulations. Thus, it is also important to contract the information in the form of simple physical concepts whenever possible.

Historically, one of the drives for the interest in simple drift-wave models such as, for instance, the Hasegawa-Mima equation [6], was indeed the earlier observations of wave-number spectra, which differed from well-known power-law predictions. The Hasegawa-Mima equation was suggested initially as a paradigm to explain the rather flat electron density fluctuation spectrum observed in the ATC tokamak using the microwave scattering [7] technique. Note that density and electrostatic potential fluctuations are equal when the electron response is adiabatic.

The canonical spectra that the simple drift-wave picture suggests (either Hasegawa-Mima or more complex reduced models), usually has the basic form $|\Phi_k|^2 \sim (\alpha + \beta k^2)^{-1}/(1+k^2)$, where α and β are coefficients and k is normalized to ρ_s^{-1} (e.g., [8]). However, this is not what was observed in numerical simulations of these models. This is believed to be due to the effect of dissipation on the stability properties of this spectrum. In fact classical dual cascade of Kraichnan and Kolmogorov [9], which corresponds to $|\Phi_k|^2 \sim k^{-14/3}$ for the inverse cascade range and $|\Phi_k|^2 \sim k^{-6}$ for the forward cascade range was more common (e.g., [10]). The dual cascade picture is a result of local interactions and the fact that energy and enstrophy are conserved by the inviscid nonlinear dynamics. Thus it can be recovered from a simple shell model of drift-wave turbulence [11–13].

During the decade that followed these earlier developments, it became evident that zonal flows, and other large scale structures such as GAMs or generalized convective cells, played an important role in both dynamics and regulation of plasma microturbulence [14]. It turned out that the disparate scale interactions responsible for the formation of sheared flows, and self-regulation of turbulence by these sheared flows were as important, dynamically, as the local interactions that resulted in the cascade.

Here we propose a simple spectral model, which is devised as a ''shell model'' with disparate scale interactions. We impose the conservation ''potential enstrophy'' as the primary constraint on dynamics in order to derive it. From our interpretation of this simple shell model, we obtain an ''isotropized'' spectrum of the form

$$|n_k|^2 \sim |\Phi_k|^2 \propto \frac{k^{-3}}{(1+k^2)^2} \qquad (1)$$

when the disparate scale interactions are dominant. Here the fluctuation spectrum is averaged over the angular variable (α_k) in Cartesian k space consisting of k_θ and k_r [i.e., $\alpha_k \equiv a\tan(k_\theta/k_r)$]. This implies that the small scale turbulence is assumed to be isotropic.

Note that even though we used the shell model to obtain (1), it in fact follows from the physical assumptions of nonlocal interactions and isotropy and is a general feature of drift-wave turbulence under these assumptions. Also, in our formulation, the large scale structures corresponding to convective cells, or zonal flows (i.e., $k_\parallel = 0$ modes) are not assumed to be isotropic. However, the small scales (i.e.,

$k_\parallel \neq 0$ modes) are assumed to respond adiabatically (i.e., sufficiently rapidly) and isotropize themselves as they are refracted by possibly anisotropic large scale flows. If this condition does not hold and the small scale turbulence deviates significantly from isotropy, neither the use of a shell model nor of (1) may be justified.

Derivation of the shell model can be started with the introduction of the conserved field, the potential vorticity (PV) [15,16], which is defined as

$$h \equiv n - \nabla^2\Phi.$$

Notice that density n in potential vorticity includes background density $n_0(x)$, density corrugations $\bar{n}(x, t)$, and the fluctuations $\tilde{n}(\mathbf{x}, t)$. For instance the Hasegawa-Wakatani system (when perpendicular viscosity is unimportant) can be described as conservation of PV:

$$\frac{dh}{dt} = 0. \tag{2}$$

Similarly a simple ion temperature gradient driven (ITG) [17] turbulence model can be written as a conservation of PV, defined as $h \equiv n - \nabla^2\Phi - P/\Gamma$, etc. (where P is pressure and Γ is the adiabaticity coefficient) [18]. We also note that PV conservation corresponds to conservation of electron density with the gyrokinetic Poisson equation in the lowest order.

Starting from (2) and expanding it into fluctuations and mean flows, we obtain

$$\frac{\partial \tilde{h}}{\partial t} + \hat{\mathbf{z}} \times \nabla\bar{\Phi} \cdot \nabla\tilde{h} = \hat{\mathbf{z}} \times \nabla h_0 \cdot \nabla\tilde{\Phi} + \hat{\mathbf{z}} \times \nabla\bar{h} \cdot \nabla\tilde{\Phi} + \hat{\mathbf{z}} \times \nabla\tilde{h} \cdot \nabla\tilde{\Phi}, \tag{3}$$

where $h_0 = h_0(x)$ is the background profile of PV whose gradient acts as a free energy for this system. Here we denote slowly evolving $k_\parallel = 0$ modes (mean fields) with $\overline{(\cdot)}$ while the fluctuations are denoted by $\widetilde{(\cdot)}$. The right-hand side of (3) corresponds to linear and nonlinear growth and/or damping, and it is necessary to describe the dynamics of $\tilde{\Phi}$ separately in order to actually obtain these quantities. The corresponding mean field equation is

$$\frac{\partial \bar{h}}{\partial t} + \langle \hat{\mathbf{z}} \times \nabla\tilde{\Phi} \cdot \nabla\tilde{h} \rangle = 0.$$

Here we consider $\nu = 0$ (no collisional damping) for simplicity. In a numerical implementation of the shell model, one needs viscous or hyperviscous damping in addition to a drag term on the zonal flow equation in order to reach steady state.

The shell model corresponding to (3) can be constructed by taking circular "shells" in k space such that each shell is described by the magnitude of its wave number $k_n = g^n k_0$ (where $g > 1$ defines the logarithmic distance among shells). The potential vorticity in the nth shell is defined as the shell variable

$$h_n = \left[2\pi \int_{k_n}^{k_{n+1}} \langle |\tilde{h}_k|^2 \rangle k dk \right]^{1/2}$$

respecting the form of the interaction coefficients and conservation of total potential enstrophy, we obtain

$$\begin{aligned} \frac{\partial}{\partial t} h_n - \gamma_n h_n - \alpha p k_n (\bar{\Phi} h_{n+1} - g^{-1}\bar{\Phi} h_{n-1}) \\ = \alpha p k_n (\bar{h}\Phi_{n+1} - g^{-1}\bar{h}\Phi_{n-1}) + C(h, \Phi), \end{aligned} \tag{4}$$

where p is the wave number of the large scale mode and α is a "free" parameter for the shell model, representing the strength of the nonlinear term due to turbulence decorrelation by large scales. Its determination for the physical system is out of scope of the current Letter. The first term on the left of (4) represents linear coupling with the background gradient $dh_0/dx \sim dn_0/dx$. The coupling term is in fact proportional to dn_0/dx and Φ_n instead of h_n. However, for simplicity we do not specify a separate Φ_n equation here and instead use a generic linear growth to represent that term. Similarly if there is collisional and/or Landau damping on fluctuations, these would also enter into the expression for γ_n.

In order to compute the terms on the right-hand side, which correspond to nonlinear growth or damping, an equation for Φ is needed. We can write such an equation for a given model (e.g., Hasegawa-Wakatani or ITG). The detailed derivation of this model will be given in a future paper, where it is shown that (4) can also be derived from a full shell model of Hasegawa-Wakatani system (with separate equations for n_n and Φ_n) by forming $h_n = n_n + k_n^2\Phi_n$. However, in practice, $h_n \sim \Phi_n(1 + k_n^2)$ may be used as a crude approximation.

The local cascade is described here by the "wave collision" operator $C(h, \Phi)$:

$$\begin{aligned} C(h, \Phi) \equiv \alpha' k_n^2 \{ & g^{-3}(\Phi_{n-2}h_{n-1} - \Phi_{n-1}h_{n-2}) \\ & - g^{-1}(\Phi_{n-1}h_{n+1} - \Phi_{n+1}h_{n-1}) \\ & + g(\Phi_{n+1}h_{n+2} - \Phi_{n+2}h_{n+1})\}. \end{aligned} \tag{5}$$

Here we note that for $h_n \sim \Phi_n(1 + k_n^2)$ one can show that $\Phi_n \sim k_n^{-4/3}$ and $\Phi_n \sim k_n^{-2}$ make $C(h, \Phi)$ vanish exactly. The two additional nontrivial solutions given by the full system are $\{\Phi_n, h_n\} \propto \{k_n^{-4/3}, k_n^{-1/3}\}$ and $\{\Phi_n, h_n\} \propto \{k_n^{-2}, k_n^0\}$, which yield $|\tilde{h}_k|^2 \propto k^{-8/3}$ and $|\tilde{h}_k|^2 \propto k^{-2}$, the latter being the so-called Batchelor's spectrum [i.e., $|\tilde{h}_k|^2 k \rightarrow F(k) \propto k^{-1}$]. Complementary to (4) is the mean PV equation

$$\frac{\partial}{\partial t}\bar{h} - \sum_n \alpha g^{-1} k_n (\Phi_n h_{n-1} - \Phi_{n-1}h_n) = 0. \tag{6}$$

Where $\bar{h} \equiv [2\pi \int_0^q \langle |\bar{h}_{q'}|^2 \rangle q' dq']^{1/2}$ and the total enstrophy is defined as $W = \bar{h}^2 + \sum_n h_n^2$, whose conservation leads to the specific form for the coefficients of Eq. (4) and sets the form of Eq. (6). Local interactions described by $C(h, \Phi)$ on the other hand, survive also in the absence of mean flows and thus are taken to conserve the total fluctuation enstrophy $\tilde{W} = \sum_n h_n^2$. However, note that the full Hasegawa-Wakatani shell model, from which (4) and (6) can be deduced has 4 nonlinearly conserved quantities. The extra conservation laws follow from the dynamics of the electrostatic field (fluctuating and mean), which are unspecified in (4) and (6) for generality.

The stationary limit of turbulence spectrum can be obtained by setting $\frac{\partial}{\partial t} \rightarrow 0$ in (4) and (6). Furthermore we drop all the terms corresponding to the right-hand side of (3) in order to obtain a spectrum dominated by nonlocal interactions with a large scale flow. In this limit, we have

$$\alpha k_n(\bar{\Phi} h_{n+1} - g^{-1}\bar{\Phi} h_{n-1}) \approx 0,$$

which can be solved as a simple power law for the PV spectrum: $h_n \sim h_0 k_n^{-1/2}$. For the near adiabatic limit of the Hasegawa-Wakatani system, $h_k \sim (1 + k^2 + i\delta_k)\Phi_k$, this becomes, $h_n \sim (1 + k_n^2)\Phi_n$ or $\Phi_n^2 \sim k_n^{-1}(1 + k_n^2)^{-2}$ for the shell variable. This implies

$$|\tilde{\Phi}_k|^2 \sim |\tilde{n}_k|^2 \sim \frac{k^{-3}}{(1 + k^2)^2}. \tag{7}$$

This is the spectrum implied by disparate scale interactions with a mean or zonal flow. Of course, these are not the "only" types of interactions and one needs to consider the more general case. However, it is a simple and interesting limiting case, whose signatures are pronounced also in numerical integrations of (4) and (6) without making the assumptions leading to (7). The fact that it compares reasonably well with experimental spectra (see Fig. 1) suggests also that it may be relevant in some cases for tokamak turbulence.

In addition, we would like to note that this spectrum is not an artifact of the shell model that we present here. It rather follows from the physical assumptions that we have made (i.e., disparate scale interactions and small scale isotropy). It can also be derived, for example, using the isotropic form of the k-space diffusion equation describing the disparate scale interactions for the "generalized" Hasegawa-Mima system [19] under these assumptions, by using the simple estimate $\tau_k^{-1} \sim \bar{v}k$ for the triad interaction time with the mean flow.

The density fluctuation k spectrum used for the comparison has been obtained in Tore Supra from scattering of

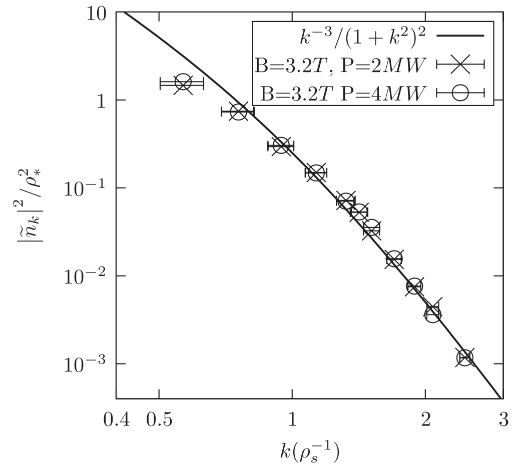

FIG. 1. A log vs log plot of density fluctuation spectrum normalized to $\rho_* = \rho_s/a$. Here while the solid line is the theoretical prediction $|\tilde{n}_k|^2 \propto k^{-3}/(1 + k^2)^2$, the crosses and circles correspond to CO_2 laser scattering data from Tore Supra with ion cyclotron resonance heated (ICRH) plasma with magnetic field $B = 3.2$ T and power $P = 2$ MW and $P = 4$ MW, respectively. It shows a reasonable agreement for $k >$ 0.6–0.7. Note that the CO_2 scattering measures primarily the binormal wave number $k_y \equiv \hat{\mathbf{b}} \times \hat{\mathbf{r}} \cdot \mathbf{k}$. Note that the error bars indicate the resolution of k only. We used an average value of ρ_s. An error in this value would simply move the whole figure sideways.

electromagnetic waves (CO_2 laser). The diagnostic [20] was based on coherent forward Thomson scattering, which is a suitable technique for the purpose of fine scale analysis, since it provides directly the space Fourier transform of the fluctuating density at a specified wave number. The diagnostic was designed for direct and detailed study of turbulence scales, with a high wavelength resolution ($\Delta k \sim 1.35$ cm^{-1}), and was used in the range $5.5 < k_\perp <$ 26 cm^{-1}. The data displayed in Fig. 1 are from L mode plasma [3], with ICRH heating and $\rho_s \approx 1$ mm.

A continuum limit of the shell model can be derived by taking $g \sim 1 + \epsilon$ and redefining $h_{k_n} \approx h_n(k_n)/k_n\sqrt{2\pi\epsilon}$ (i.e., we divide by the shell "volume"). Substituting $h_n \rightarrow kh(k)\sqrt{\epsilon}$ and expanding in ϵ:

$$\frac{\partial h(k)}{\partial t} + 2\alpha\epsilon p\bar{\Phi}k^{1/2}\frac{\partial}{\partial k}(k^{3/2}h(k)) = \gamma(k)h(k) + 2\alpha\epsilon p\bar{h}\left(k^{1/2}\frac{\partial}{\partial k}(k^{3/2}\Phi(k))\right) + \frac{C(\Phi, h)}{k\sqrt{\epsilon}}, \tag{8}$$

with the collision term

$$C(\Phi, h) \approx \alpha' k^2\epsilon^4\left\{2k^5\left[\Phi(k)^{-1/2}\frac{d}{dk}\left(\frac{d^2}{dk^2}h(k)\Phi(k)^{3/2}\right) - h(k)^{-1/2}\frac{d}{dk}\left(\frac{d^2}{dk^2}\Phi(k)h(k)^{3/2}\right)\right]\right.$$
$$\left. + 21k^2\frac{d}{dk}\left(\left[\frac{dh(k)}{dk}\Phi(k) - \frac{d\Phi(k)}{dk}h(k)\right]k^2\right)\right\}. \tag{9}$$

It can easily be verified that nontrivial power-law solutions that make (5) vanish, such as $\Phi(k) \propto k^{-7/3}$ and $h(k) \propto k^{-4/3}$, make (9) vanish also. Similarly, $h(k) \propto k^{-3/2}$, make the $\frac{\partial}{\partial k}(k^{3/2}h(k))$ term in (8) vanish. Multiplying (8) by $2h(k)$, we obtain

$$\frac{\partial h(k)^2}{\partial t} + \frac{1}{k}\frac{\partial}{\partial k}(k[2\alpha\epsilon(p\bar{\Phi})(h(k)^2k^2)]) = 2\gamma(k)h(k)^2 + 4\alpha\epsilon p\bar{h}k^{1/2}h(k)\frac{\partial}{\partial k}(k^{3/2}\Phi(k)) + \frac{2h(k)C(\Phi, h)}{k\sqrt{\epsilon}}, \tag{10}$$

where the left-hand side corresponds to the $\frac{\partial}{\partial t} W_k + \nabla_k \cdot (\bar{\mathbf{V}} k^2 W_k)$ in contrast to the wave-kinetic equation [21,22]:

$$\frac{\partial}{\partial t} N_{\mathbf{k}}(t) + \nabla_k \cdot \left\{ \int\!\!\int d\mathbf{x} [\bar{\omega}_k(x) \nabla N_{\mathbf{k}}(\mathbf{x}, t)] \right\} = C, \qquad (11)$$

where $\bar{\omega}_k(x) \approx \bar{V}(x) k_y$. Going from (11) to the corresponding terms in (10) requires the nontrivial assumptions of isotropy (i.e., $k_x \sim k_y$) and scale separation [i.e., $N_k(x)^{-1} \partial_X N_k(x) \sim \epsilon k$]. Note also the appearance of the cylindrical divergence operator in (11), since we use $k = |\mathbf{k}|$ as the independent variable.

In order to write the equation for energy, we multiply (8) by $\Phi(k)$ instead of $h(k)$ and use $h(k) = (1 + k^2)\Phi(k)$. This allows us to contrast the collision operator in (10), as given in (9), with the differential approximation models used in fluid dynamics [23,24]. Substituting $h(k) = (1 + k^2)\Phi(k)$ into (9) and rearranging, we obtain

$$C[\Phi, (1 + k^2)\Phi] \rightarrow \frac{3\alpha' \epsilon^4}{\Phi(k)} \frac{\partial}{\partial k} \left(k^2 \Phi(k) \frac{\partial}{\partial k} (k^6 \Phi(k)^2) \right),$$

which is exactly the same as in Refs. [23,24], when written in terms of $\Phi(k)$.

The continuum limit can be useful in application to the dissipative range. Assume that the "effective" linear growth or damping in steady state is given by $\gamma_R(k)$ as a function of k, where the steady state can be described as a balance between the second term on the left and the first term on the right-hand side of (8):

$$2\alpha \epsilon p \bar{\Phi} k^{1/2} \frac{\partial}{\partial k} (k^{3/2} h(k)) = \gamma_R(k) h(k). \qquad (12)$$

Here the right-hand side is taken to represent the difference between growth due to $h_0 + \bar{h}$, and the damping, including eddy damping. At small scales, $\gamma_R(k)$ becomes negative, due to dissipation, eddy damping or landau damping dominating over linear growth. If we take $\gamma_R(k) \sim -\lambda_R k^2$, representing this "dissipative range," we obtain for the fluctuating density that

$$\langle |\tilde{n}_k|^2 \rangle \sim \langle |\tilde{\Phi}_k|^2 \rangle \sim \frac{k^{-3}}{(1 + k^2)^2} e^{-\lambda k},$$

where $\lambda \sim \lambda_R / 2\alpha \epsilon p \bar{\Phi}$, which suggests that the spectrum is affected by the growth-damping profile.

We developed a simple shell model for the evolution of turbulence spectra, which includes local and disparate scale interactions in addition to linear growth or damping. The derivation is based on PV and enstrophy conservation, while the local interactions conserve fluctuation enstrophy disparate scale interactions conserve total (fluctuation + mean) enstrophy. We observed that the same model can be derived using a 2-field system (such as Hasegawa-Wakatani system) and combining the separate equations for density and electrostatic field in order to describe PV evolution. Using this model, when the local interactions are dominant, the Kraichnan-Kolmogorov picture of dual cascade can be recovered, whereas if disparate scale interactions are dominant, an isotropic small scale spectrum of the form $k^{-3}/(1 + k^2)^2$ was obtained. We observed that this simple analytical form compares reasonably well with the density fluctuation spectrum from Tore Supra tokamak measured using CO_2 laser scattering.

We would like to thank S. Nazarenko and M. Ottaviani for valuable discussions. This work, supported by the European Communities under the contract of Association between EURATOM, CEA, and the French Research Federation for fusion studies, was carried out within the framework of the European Fusion Development Agreement. Financial support was also received from the Agence Nationale de la Recherche under Contract No. 06-3_134975. The views and opinions expressed herein do not necessarily reflect those of the European Commission.

*ozgur.gurcan@cea.fr

[1] R. J. Fonck *et al.*, Phys. Rev. Lett. **70**, 3736 (1993).
[2] G. McKee *et al.*, Rev. Sci. Instrum. **70**, 913 (1999).
[3] P. Hennequin *et al.*, Plasma Phys. Controlled Fusion **46**, B121 (2004).
[4] S. Zweben *et al.*, Nucl. Fusion **44**, 134 (2004).
[5] P. W. Terry *et al.*, Phys. Plasmas **15**, 062503 (2008).
[6] A. Hasegawa and K. Mima, Phys. Fluids **21**, 87 (1978).
[7] E. Mazzucato, Phys. Rev. Lett. **36**, 792 (1976).
[8] A. E. Koniges *et al.*, Phys. Fluids B **3**, 1297 (1991).
[9] R. H. Kraichnan, Phys. Fluids **10**, 1417 (1967).
[10] D. Fyfe and D. Montgomery, Phys. Fluids **22**, 246 (1979).
[11] M. Yamada and K. Ohkitani, Prog. Theor. Phys. **79**, 1265 (1988).
[12] R. Benzi, L. Biferale, and G. Parisi, Physica (Amsterdam) **65D**, 163 (1993).
[13] J. L. Ottinger and D. Carati, Phys. Rev. E **48**, 2955 (1993).
[14] P. H. Diamond, S.-I. Itoh, K. Itoh, and T. S. Hahm, Plasma Phys. Controlled Fusion **47**, R35 (2005).
[15] P. B. Rhines and W. R. Young, J. Fluid Mech. **122**, 347 (1982).
[16] P. H. Diamond, O. D. Gurcan, T. S. Hahm, K. Miki, Y. Kosuga, and X. Garbet, Plasma Phys. Controlled Fusion **50**, 124018 (2008).
[17] B. Coppi, M. N. Rosenbluth, and R. Z. Sagdeev, Phys. Fluids **10**, 582 (1967).
[18] A. I. Smolyakov, P. H. Diamond, and M. V. Medvedev, Phys. Plasmas **7**, 3987 (2000).
[19] J. A. Krommes and C. B. Kim, Phys. Rev. E **62**, 8508 (2000).
[20] A. Truc *et al.*, Rev. Sci. Instrum. **63**, 3716 (1992).
[21] B. Dubrulle and S. Nazarenko, Physica (Amsterdam) **110D**, 123 (1997).
[22] A. I. Smolyakov and P. H. Diamond, Phys. Plasmas **6**, 4410 (1999).
[23] V. S. L'vov and S. Nazarenko, JETP Lett. **83**, 541 (2006).
[24] D. K. Lilly, J. Atmos. Sci. **46**, 2026 (1989).